Generalized connected sum construction for constant scalar curvature metrics

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Abstract

In this paper we construct constant scalar curvature metrics on the generalized connected sum $M = M_1 \sharp_K M_2$ of two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) along a common Riemannian submanifold (K, g_K), in the case where the codimension of K is ≥ 3 and the manifolds M_1 and M_2 carry the same nonzero constant scalar curvature S. In particular the structure of the metrics we build is investigated and described.

1 Introduction and statement of the result

Connected sum of solutions of nonlinear problems has revealed to be a very powerful tool in understanding solutions of many geometric problems (minimal and constant mean curvature surfaces [7], [8], constant scalar curvature metrics [4], [9], [6], and recently even Einstein metrics [1]). However, generalized connected sums along a submanifold have not been addressed so much, probably because these constructions are less flexible.

In this paper we consider the problem of constructing solutions to the Yamabe equation (i.e. conformal constant scalar curvature metrics) on the generalized connected sum $M = M_1 \sharp_K M_2$ of two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) along a common (isometrically embedded) submanifold (K, g_K) of codimension ≥ 3. We are able to perform this generalized connected sum under the assumptions that the two initial Riemannian metrics have the same constant scalar curvature S and the linearized Yamabe operator about the metrics g_i (i.e. the operators $\Delta_{g_i} + S/(n-1)$) have trivial kernels, for $i = 1, 2$.

To put this result in perspective, let us recall the classical result of Schoen-Yau [11] and Gromov-Lawson [10] which ensures that if the manifolds M_1 and

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M_2 carry positive scalar curvature metrics, then so does the generalized connected sum $M = M_1 \sharp_K M_2$ along a submanifold K of codimension ≥ 3 and, thanks to the resolution of the Yamabe problem by T. Aubin and R. Schoen, M can be endowed with a constant positive scalar curvature metric. This result however does not give the precise structure of the constant scalar curvature metric one obtains on the generalized connected sum M. In particular, one would like to know how does the constant scalar curvature metric on the connected sum looks like in terms of the constant scalar curvature metric on the summands. Our result does not cover all cases covered by the above mentioned result but, as it is typical for most of the gluing results, we have a very precise description of the metric on the connected sum in terms of the metric on the summands. Indeed, away from the region where the generalized connected sum takes place, we obtain metrics on M which are conformal to the metrics g_i with some conformal factor as close to the constant function 1 as we want.

In the case of connected sum at points a result analogous to ours had been obtained by D. Joyce [4]. Our strategy is roughly speaking the same : we first write down a one dimensional family of approximate solutions metrics $(g_\varepsilon)_{\varepsilon\in(0,1)}$ (where the parameter ε represent the size of the tubular neighborhood we excise from each manifold in order to perform the generalized connected sum), then, we study the linearized scalar curvature operator about the metric g_ε and, for all sufficiently small ε, we find suitable conformal factors u_ε such that the metrics $\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} g_\varepsilon$ have constant scalar curvature S using a simple fixed point argument. Let us now describe our result more precisely.

Let (M_1, g_1) and (M_2, g_2) be two m-dimensional compact Riemannian manifolds with constant scalar curvature S, and suppose that there exists a k-dimensional Riemannian manifold (K, g_K) which is isometrically embedded in each (M_i, g_i), for $i = 1, 2$, $m \geq 3$, $m - k \geq 3$. We also assume that the normal bundles of K in (M_i, g_i) can be diffeomorphically identified. Finally, we assume that on both manifolds, the operator

$$L_{g_i} := \Delta_{g_i} + \frac{S}{n-1}$$

is injective.

Let $M = M_1 \sharp_K M_2$ be the generalized connected sum of (M_1, g_1) and (M_2, g_2) along K which is obtained by removing an ε-tubular neighborhood of K from each M_i and identifying the two boundaries.

Our main result reads :

Theorem 1.1. *Under the above assumptions, it is possible to endow M with a family of constant scalar curvature metrics $\tilde{g}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0)$ whose scalar curvature $S_{\tilde{g}_\varepsilon}$ is constant equal to S. In addition, the following holds*

(i) - The metric $\tilde{g}_\varepsilon$ is conformal to the metrics g_i away from a fixed (small) tubular neighborhood of K in M_i, $i = 1, 2$ for a conformal factor u_ε which can

be chosen so that

$$\|u_\varepsilon - 1\|_{L^\infty(M)} \leq c\,\varepsilon^{\frac{n-2}{2}-\delta},$$

where $\max\{0,(n-4)/2\} < \delta < (n-2)/2$, $n = m-k$ *and* $c > 0$ *does not depend on* ε.

(ii) - As ε *tends to* 0*, the metrics* $\tilde{g}_\varepsilon$ *converge to* g_i *on compacts of* $M_i \backslash \iota_i(K)$*,* $i = 1, 2$.

A typical case where our result applies is when both $(M_1, g_1) = (M_2, g_2)$ and K is any submanifold of codimension ≥ 3, provided the operator L_{g_i} has no nontrivial kernel.

There are some main technical differences between our construction and D. Joyce's construction in the connected sum case. Our construction seems to be less flexible in the sense that more hypothesis are needed on the summands to obtain the result. In particular (so far) the construction only holds when (K, g_K) is isometrically embedded in both (M_i, g_i) and if this is not the case it seems harder to construct a reasonable approximate solution g_ε to our problem. The second difference comes from the analysis of the operator L_{g_ε}, the linearized scalar curvature operator about the metric g_ε. As in the connected sum case, the derivation of the estimates of the solution of $L_{g_\varepsilon}\, u = f$ follows from application of the maximum principle. However, in the generalized connected sum case, the estimates for the partial derivatives of the solution u are not as nicely behaved as in the connected sum case. Hopefully, the scalar curvature equation is a semilinear elliptic equation and hence, the nonlinear part of this equation only involves the function u and not its partial derivatives.

It is possible to extend our result to the case where $S = 0$ relaxing the fact that the scalar curvature one obtains on the summand is equal to 0. Indeed, in this case, the scalar curvature obtained on M might not be equal to 0 but will be a constant close to 0.

2 Building the metrics

Let (K, g_K) be a k-dimensional Riemannian manifold isometrically embedded in both the n-dimensional Riemannian manifolds (M_1, g_1) and (M_2, g_2),

$$\iota_i : K \hookrightarrow M_i$$

We assume that the isometric map $\iota_1^{-1} \circ \iota_2 : \iota_1(K) \to \iota_2(K)$ extends to a diffeomorphism between the normal bundles of $\iota_i(K)$ in (M_i, g_i), $i = 1, 2$. We further assume that the metrics g_1 and g_2 have the same constant scalar curvature S. In this section our aim is to perform a generalized connected sum of (M_1, g_1) and (M_2, g_2) along (K, g_K) and to construct on the new manifold $M = M_1 \sharp_K M_2$ a family of metrics $(g_\varepsilon)_{\varepsilon \in (0,1)}$, whose scalar curvature is close to S.

For a fixed $\varepsilon \in (0,1)$, we describe the generalized connected sum construction and the definition of the metric g_ε in local coordinates, the fact that this construction yields a globally defined metric will follow at once.

Let U^k be an open set of $\mathbb{R}^k$, B^{m-k} the $(m-k)$-dimensional open ball $(m-k \geq 3)$. For $i = 1, 2$, $F_i : U^k \times B^{m-k} \to W_i \subset M_i$ given by

$$F_i(z, x) := \exp_z^{M_i}(x)$$

defines local Fermi coordinates near the coordinate patches $F_i(\cdot, 0)\,(U) \subset \iota_i(K) \subset M_i$. In these coordinates, the metric g_i can be decomposed as

$$g_i(z,x) \;=\; g^{(i)}_{j_1 j_2} dz^{j_1} \otimes dz^{j_2} + g^{(i)}_{\alpha\beta} dx^\alpha \otimes dx^\beta + g^{(i)}_{j\alpha} dz^j \otimes dx^\alpha$$

and it is well known that in this coordinate system

$$g^{(i)}_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \qquad \text{and} \qquad g^{(i)}_{j\alpha} = \mathcal{O}\left(|x|\right)$$

In order to perform the identification between W_1 and W_2 and in order to glue the metrics together and define g_ε, we partially change the coordinate system, by setting

$$x = \varepsilon\, \mathrm{e}^{-t}\, \theta$$

on $F_1^{-1}(W_1)$ and

$$x = \varepsilon\, \mathrm{e}^{t} \theta$$

on $F_2^{-1}(W_2)$, for $\varepsilon \in (0,1)$, $\log\varepsilon < t < -\log\varepsilon$, $\theta \in S^{m-k-1}$.

Using these changes of coordinates the expressions of the two metrics g_1 and g_2 on $U^k \times A^1_{\varepsilon^2}$, where $A^1_{\varepsilon^2}$ is the annulus $\{\varepsilon^2 < |x| < 1\}$ become respectively

$$\begin{aligned} g_1(z,t,\theta) \;&=\; g^{(1)}_{ij} dz^i \otimes dz^j \\ &+\; u_\varepsilon^{(1)\,\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(1)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu\right) + g^{(1)}_{t\theta}\, dt \ltimes d\theta\right] \\ &+\; g^{(1)}_{it} dz^i \otimes dt + g^{(1)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned}$$

and

$$\begin{aligned} g_2(z,t,\theta) \;&=\; g^{(2)}_{ij} dz^i \otimes dz^j \\ &+\; u_\varepsilon^{(2)\,\frac{4}{n-2}} \left[\left(dt \otimes dt + g^{(2)}_{\lambda\mu} d\theta^\lambda \otimes d\theta^\mu\right) + g^{(2)}_{t\theta}\, dt \ltimes d\theta\right] \\ &+\; g^{(2)}_{it} dz^i \otimes dt + g^{(2)}_{i\lambda} dz^i \otimes d\theta^\lambda \end{aligned}$$

where by the compact notation $g_{t\theta}\, dt \ltimes d\theta$ we indicate the general component of the normal metric tensor (that is, it involves $dt \otimes dt$, $d\theta^\lambda \otimes d\theta^\mu$ and $dt \otimes d\theta^\lambda$

components).

Remark that for $j = 1, 2$ we have

$$\begin{aligned} g^{(j)}_{\lambda\mu} &= \mathcal{O}(1) & \qquad g^{(j)}_{t\theta} &= \mathcal{O}\left(|x|^2\right) \\ g^{(j)}_{it} &= \mathcal{O}\left(|x|^2\right) & \qquad g^{(j)}_{i\lambda} &= \mathcal{O}\left(|x|^2\right) \end{aligned}$$

and

$$u^{(1)}_\varepsilon(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{-\frac{n-2}{2}t} \qquad \text{and} \qquad u^{(2)}_\varepsilon(t) = \varepsilon^{\frac{n-2}{2}} \mathrm{e}^{\frac{n-2}{2}t}$$

We choose a cut-off function $\chi : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -1]$ and 0 in $[1, -\log\varepsilon)$ and we choose another cut-off function $\eta : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ to be a non increasing smooth function which is identically equal to 1 in $(\log\varepsilon, -\log\varepsilon - 1]$ and which satisfies $\lim_{t\to -\log\varepsilon} \eta = 0$. Using these two cut-off functions, we can define a new normal conformal factor u_ε by

$$u_\varepsilon(t) := \eta(t)\, u^{(1)}_\varepsilon(t) + \eta(-t)\, u^{(2)}_\varepsilon(t)$$

and the metric g_ε by

$$\begin{aligned} g_\varepsilon(z,t,\theta) \;:=\; & \left(\chi g^{(1)}_{ij} + (1-\chi) g^{(2)}_{ij}\right) dz^i \otimes dz^j \\ + \;& u_\varepsilon^{\frac{4}{n-2}} \Big[dt \otimes dt + \left(\chi g^{(1)}_{\lambda\mu} + (1-\chi) g^{(2)}_{\lambda\mu}\right) d\theta^\lambda \otimes d\theta^\mu \\ & \qquad + \left(\chi g^{(1)}_{t\theta} + (1-\chi) g^{(2)}_{t\theta}\right) dt \ltimes d\theta \Big] \\ + \;& \left(\chi g^{(1)}_{it} + (1-\chi) g^{(2)}_{it}\right) dz^i \otimes dt \\ + \;& \left(\chi g^{(1)}_{i\lambda} + (1-\chi) g^{(2)}_{i\lambda}\right) dz^i \otimes d\theta^\lambda \end{aligned} \tag{1}$$

Closer inspection of this expression shows that the only objects that are not *a priori* globally defined on the identification of the tubular neighborhoods of $\iota_1(K)$ in M_1 and $\iota_2(K)$ in M_2 are the functions χ and u_ε (since η is used in the construction). However, observe that both cut-off functions can easily be expressed as functions of the Riemannian distance to K in the respective manifolds. Hence they are globally defined and the metric g_ε - whose definition can be obviously completed by putting $g_\varepsilon \equiv g_1$ and $g_\varepsilon \equiv g_2$ out of the "polyneck" - is a Riemannian metric which is globally defined on the manifold M.

3 Estimate of the scalar curvature

Now we want to estimate the difference $S_{g_\varepsilon} - S$ on the "polyneck" (which, in the above coordinates, corresponds to $\log\varepsilon + 1 \leq t \leq -\log\varepsilon - 1$). To begin

with, we restrict our attention to the case where $\log\varepsilon \leq t \leq -1$. Here the normal conformal factor can be written down as $u_\varepsilon = u_\varepsilon^{(1)}\left(1 + u_\varepsilon^{(2)}/u_\varepsilon^{(1)}\right)$ so, if we define $h = u_\varepsilon^{(2)}/u_\varepsilon^{(1)}$ the metric g_ε looks like

$$g_\varepsilon(z,t,\theta) = g_{ij}^{(1)} dz^i \otimes dz^j + (1+h)^{\frac{4}{n-2}}\, g_{\alpha\beta}^{(1)} dx^\alpha \otimes dx^\beta + g_{i\alpha}^{(1)} dz^i \otimes dx^\alpha$$

where in fact $h = \mathrm{e}^{(n-2)t} = \varepsilon^{(n-2)}|x|^{2-n}$.

In order to simplify the notations, let us drop the upper $^{(1)}$ indices and simply write

$$g(z,x,h) \quad = \quad g_{ij} dz^i \otimes dz^j + (1+h)^{\frac{4}{n-2}}\, g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{i\alpha} dz^i \otimes dx^\alpha$$

Recall that the following expansions hold

$$\begin{aligned} g_{ij} &= g_{ij}^K(z) + \mathcal{O}(|x|) \\ g_{\alpha\beta} &= \delta_{\alpha\beta} + \mathcal{O}\left(|x|^2\right) \\ g_{i\alpha} &= \mathcal{O}(|x|) \end{aligned}$$

In the following computation we will use the notations

$$\begin{aligned} g_h(z,x) &:= g(z,x,h) \\ g_0(z,x) &:= g(z,x,0) \\ \tilde{g}_h(z) &:= g(z,0,h) \\ \tilde{g}_0(z) &:= g(z,0,0) \end{aligned}$$

and their respective scalar curvature will be denoted by

$$\begin{aligned} S_h &:= S_{g_h} \\ S_0 &:= S_{g_0} \\ \tilde{S}_h &:= S_{\tilde{g}_h} \\ \tilde{S}_0 &:= S_{\tilde{g}_0} \end{aligned}$$

The idea is to estimate the difference between the scalar curvatures of the metrics g_h and g_0 by first estimating the differences with the scalar curvature of the Riemannian product metrics $\tilde{g}_h$ and $\tilde{g}_0$. In fact, we can easily obtain

$$\tilde{S}_h \quad = \quad \tilde{S}_0 + (1+h)^{\frac{4}{n-2}}\, \Delta_{eucl}^{(x)} h$$

Next we consider the term $S_h - \tilde{S}_h$. To keep notations short, we agree that $A_k^{(j)} = A_l^{(j)}(z,x,h)$, $j,l \in \mathbb{N}$ is a function, a row vector or a matrix whose coefficients satisfy

$$\begin{aligned} \left|A_l^{(j)}(z,x,h)\right| &\leq C\,|x|^l \\ \left|A_l^{(j)}(z,x,h) - A_k^{(j)}(z,x,0)\right| &\leq C\,|x|^l\,|h| \end{aligned}$$

for some positive constant $C = C(j)$.

We start with the expansions of the coefficients of the metrics g_h (and hence also g_0 which corresponds to g_h when $h = 0$) and their inverses in terms of $|x|$

$$
\begin{aligned}
g_{ij}^{(h)} &= \tilde{g}_{ij}^{(h)}(z) + \mathcal{O}(|x|) \\
g_{\alpha\beta}^{(h)} &= \tilde{g}_{\alpha\beta}^{(h)} + \mathcal{O}\left(|x|^2\right) \\
g_{i\alpha}^{(h)} &= \mathcal{O}(|x|)
\end{aligned}
$$

and

$$
\begin{aligned}
g^{ij}_{(h)} &= \tilde{g}^{ij}_{(h)}(z) + A_1^{(1)} \\
g^{\alpha\beta}_{(h)} &= \tilde{g}^{\alpha\beta}_{(h)} + A_1^{(2)} \\
g^{i\alpha}_{(h)} &= A_1^{(3)}
\end{aligned}
$$

We estimate the Christoffel symbols of the metric g_h. Observe that

$$
\begin{aligned}
g^{\cdots}_{(h)} \frac{\partial g^{(h)}_{\cdots}}{\partial \ldots} &= \left(\tilde{g}^{\cdots}_{(h)} + A_1^{(4)}\right)\left(\frac{\partial \tilde{g}^{(h)}_{\cdots}}{\partial \ldots} + A^{(5)} + A_1^{(6)}[\nabla h]\right) \\
&= \tilde{g}^{\cdots}_{(h)} \frac{\partial \tilde{g}^{(h)}_{\cdots}}{\partial \ldots} + A_0^{(7)} + A_1^{(8)}[\nabla h]
\end{aligned}
$$

As a consequence we have that

$$
\Gamma(h, \nabla h) = \tilde{\Gamma}(h, \nabla h) + A_0^{(9)} + A_1^{(10)}[\nabla h]
$$

Moreover, it is straightforward to check that

$$
\tilde{\Gamma}(h, \nabla h) = A_0^{(11)} + A_0^{(12)}[\nabla h]
$$

Proceeding with the computation we get

$$
\frac{\partial \Gamma}{\partial \ldots}(h, \nabla h) = \frac{\partial \tilde{\Gamma}}{\partial \ldots}(h, \nabla h) + A_0^{(13)}[\nabla h] + A_1^{(14)}[\nabla h, \nabla h] + A_1^{(15)}\left[\nabla^2 h\right]
$$

$$
\frac{\partial \tilde{\Gamma}}{\partial \ldots}(h, \nabla h) = A_0^{(16)}[\nabla h] + A_0^{(17)}[\nabla h, \nabla h] + A_0^{(18)}\left[\nabla^2 h\right]
$$

while for the product of Christoffel symbols, we get

$$
\Gamma\,\Gamma(h, \nabla h) = \tilde{\Gamma}\,\tilde{\Gamma}(h, \nabla h) + A_0^{(19)} + A_0^{(20)}[\nabla h] + A_1^{(21)}[\nabla h, \nabla h]
$$

and hence we get for the coefficients of the curvature tensors

$$
\begin{aligned}
R(h, \nabla h, \nabla^2 h) &= \tilde{R}(h, \nabla h, \nabla^2 h) + A_0^{(22)} + A_0^{(23)}[\nabla h] \\
&+ A_1^{(24)}[\nabla h, \nabla h] + A_1^{(25)}\left[\nabla^2 h\right]
\end{aligned}
$$

$$
\tilde{R}(h, \nabla h, \nabla^2 h) = A_0^{(26)} + A_0^{(27)}[\nabla h] + A_0^{(28)}[\nabla h, \nabla h] + A_0^{(29)}\left[\nabla^2 h\right]
$$

Finally, observing that

$$g_h^{\cdot\cdot}g_h^{\cdot\cdot} \;=\; \tilde{g}_h^{\cdot\cdot}\tilde{g}_h^{\cdot\cdot} + A_1^{(30)}$$

and contracting twice the Riemann tensor, we get the expression for the scalar curvature

$$S_h \;=\; \tilde{S}_h + A_0^{(31)} + A_0^{(32)}\left[\nabla h\right] + A_1^{(33)}\left[\nabla h, \nabla h\right] + A_1^{(34)}\left[\nabla^2 h\right]$$

Choosing $h \equiv 0$ in the previous computation we obtain immediately

$$S_0 \;=\; \tilde{S}_0 + A_0^{(35)}(z,x,0)$$

Hence we have obtained

$$\begin{aligned} S_h \;=\; S_0 \;&+\; (1+h)^{-\frac{n+2}{n-2}}\Delta_{eucl}^{(x)}h + A_0^{(36)}(z,x,h) - A_0^{(36)}(z,x,0) \\ &+\; A_0^{(37)}\left[\nabla h\right] + A_1^{(38)}\left[\nabla h, \nabla h\right] + A_1^{(39)}\left[\nabla^2 h\right] \end{aligned}$$

Since $h = \varepsilon^{n-2}|x|^{2-n}$ is $\Delta_{eucl}^{(x)}$-harmonic we conclude that

$$\begin{aligned} S_h - S_0 &= A_0^{(40)} + A_0^{(41)}[\nabla h] + A_1^{(42)}[\nabla h, \nabla h] + A_1^{(43)}[\nabla^2 h] \\ &= \mathcal{O}\left(\varepsilon^{n-2}|x|^{1-n}\right) \\ &= \mathcal{O}\left(\varepsilon^{-1}\mathrm{e}^{(n-1)t}\right) \end{aligned}$$

We remark that, when $t = \log\varepsilon + 1$, we get the estimate $S_{g_\varepsilon} - S_{g_1} = \mathcal{O}(\varepsilon^{n-2})$.

Let us now treat the case where $-1 \leq t \leq 0$. The action of the cut-off function is effective here, so *a priori* we have to handle the full expression of g_ε. In any case, it is easy to see that one can always write for $-1 \leq t \leq 0$

$$\begin{aligned} g_\varepsilon(z,t,\theta) \;=\; &\left(g_{ij}^1 + \mathcal{O}\left(|x|\right)\right) dz^i \otimes dz^j \\ +\; &(1+h)^{\frac{4}{n-2}}\left(g_{\alpha\beta}^{(1)} + \mathcal{O}\left(|x|\right)\right) dx^\alpha \otimes dx^\beta \\ +\; &\left(g_{i\alpha}^{(1)} + \mathcal{O}\left(|x|\right)\right) dz^i \otimes dx^\alpha \end{aligned}$$

Hence, if we take $g(z,x,h) = g_\varepsilon$ and $g(z,x,0) = g_1 + \mathcal{O}\left(|x|\right)$ in the previous computation we get immediately $S_{g_\varepsilon} - S_{g_1+\mathcal{O}(|x|)} = \mathcal{O}\left(\varepsilon^{n-2}|x|^{1-n}\right)$.

Now we observe that in general if we have two metrics g and $\hat{g}$ such that $\hat{g} = g + \mathcal{O}\left(|x|\right)$, then $\hat{\Gamma} = \Gamma + \mathcal{O}\left(1\right)$ and $\hat{R} = R + \mathcal{O}\left(|x|^{-1}\right)$, so the scalar curvatures of g and $\hat{g}$ are related by $\hat{S} = S + \mathcal{O}\left(|x|^{-1}\right)$.

To conclude, we have that

$$S_{g_\varepsilon} - S_{g_1} = \mathcal{O}\left(|x|^{-1}\right) = \mathcal{O}\left(\varepsilon^{-1}\mathrm{e}^{t}\right)$$

for $-1 \leq t \leq 0$. In particular, when $t = 0$ we get $S_{g_\varepsilon} - S_{g_1} = \mathcal{O}\left(\varepsilon^{-1}\right)$. Similar estimates hold for $S_{g_\varepsilon} - S_{g_2}$ when $0 \leq t \leq -\log\varepsilon - 1$ and hence we have obtained the

Lemma 3.1. *There exists a constant $c > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$|S_{g_\varepsilon} - S| \leq c\,\varepsilon^{-1}\,(\operatorname{ch} t)^{1-n}$$

for $|t| \leq |\log\varepsilon| - 1$.

4 Analysis of a linear operator

In order to obtain the proof of the main Theorem, we want to solve, using a perturbation argument, the Yamabe equation

$$\Delta_{g_\varepsilon} u + c_n S_{g_\varepsilon} u \quad = \quad c_n S u^{\frac{n+2}{n-2}} \tag{2}$$

where $c_n = -(n-2)/4(n-1)$.

If we are able to find such a function u, then, by performing the conformal change $\tilde{g}_\varepsilon = u^{\frac{4}{n-2}} g_\varepsilon$ we get a metric $\tilde{g}_\varepsilon$, whose scalar curvature is the constant equal to S.

We write $u = 1 + v$ where v is a small function ($|v| \leq 1/2$) so that the equation becomes

$$\begin{aligned} \Delta_{g_\varepsilon} v - \frac{4c_n}{n-2} S_{g_\varepsilon} v \quad &= \quad c_n\,(S - S_{g_\varepsilon}) + c_n \frac{n+2}{n-2}\,(S - S_{g_\varepsilon})\,v \\ &+ \quad c_n S\left((1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2} v\right) \end{aligned} \tag{3}$$

We define the linearized scalar curvature operator by

$$L_{g_\varepsilon} := \Delta_{g_\varepsilon} - \frac{4c_n}{n-2} S_{g_\varepsilon} = \Delta_{g_\varepsilon} + \frac{S_{g_\varepsilon}}{n-1}$$

Our aim is to study the operator L_{g_ε} and provide an *a priori* estimate for the solutions of the linear problem

$$L_{g_\varepsilon} v = f$$

This is the starting point and the key-tool for the nonlinear perturbation argument.

Unfortunately a global *a priori* estimate is not immediately available. We will be able to obtain such an estimate using an argument by contradiction, once a local *a priori* estimate is obtained for the solutions of the linearized problem on the "polyneck".

4.1 Local expression for Δ_{g_ε} on the "polyneck" and barrier functions

The first step is to write down the local expression for the g_ε-laplacian, which is the principal part of our operator, on the "polyneck". Clearly, we can restrict ourselves to the set $\{\log\varepsilon + 1 \le t \le 0\}$ where $|x| = \varepsilon \mathrm{e}^{-t}$. We have at hand the expansions

$$\begin{aligned}
g^\varepsilon_{ij} &= g^K_{ij}(z) + \mathcal{O}(|x|) \\
g^\varepsilon_{it} &= \mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{i\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{tt} &= u_\varepsilon^{\frac{4}{n-2}}\left(1 + \mathcal{O}\left(|x|^2\right)\right) \\
g^\varepsilon_{t\lambda} &= u_\varepsilon^{\frac{4}{n-2}}\mathcal{O}\left(|x|^2\right) \\
g^\varepsilon_{\lambda\mu} &= u_\varepsilon^{\frac{4}{n-2}}\left(g_{\lambda\mu}(\theta) + \mathcal{O}\left(|x|^2\right)\right)
\end{aligned}$$

where $g_{\lambda\mu}(\theta)$ is the common value of $g^{(1)}_{\lambda\mu}(\theta)$ and $g^{(2)}_{\lambda\mu}(\theta)$. Hence

$$\sqrt{g_\varepsilon} \quad = \quad \sqrt{\det(g^K_{ij}(z))}\,\sqrt{\det(g_{\lambda\mu}(z))}\; u_\varepsilon^{\frac{2n}{n-2}}(t)\;\left[1 + \mathcal{O}(|x|)\right]$$

So, for coefficients of the inverse matrix we have the expansions

$$\begin{aligned}
g_\varepsilon^{ij} &= g_K^{ij}(z) + \mathcal{O}(|x|) \\
g_\varepsilon^{it} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{i\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{tt} &= u_\varepsilon^{-\frac{4}{n-2}}\left[1 + \mathcal{O}(|x|)\right] \\
g_\varepsilon^{t\lambda} &= \mathcal{O}\left(|x|^2\right) \\
g_\varepsilon^{\lambda\mu} &= u_\varepsilon^{-\frac{4}{n-2}} g^{\lambda\mu}\left[1 + \mathcal{O}(|x|)\right]
\end{aligned}$$

A straightforward computation yields the expression we were looking for

$$\Delta_{g_\varepsilon} = u_\varepsilon^{-\frac{4}{n-2}}\left[\partial_t^2 + (n-2)\,\mathrm{th}\left(\frac{n-2}{2}t\right)\partial_t + \Delta^{(\theta)}_{S^{n-1}} + u_\varepsilon^{\frac{4}{n-2}}\Delta^{(z)}_K + \mathcal{O}(|x|)\,\Phi(\nabla,\nabla^2)\right] \tag{4}$$

where $\Phi(\nabla,\nabla^2)$ is a nonlinear differential operator involving first order and second order partial derivatives with respect to t, θ^λ and z^j and whose coefficients are bounded uniformly on the "polyneck", as $\varepsilon \in (0,1)$.

To obtain the local *a priori* estimates, the key tool is the maximum principle for the g_ε-Laplacian and the construction of barrier functions. In order to find the later, let us remark that

$$\left(\partial_t^2 + \left(\frac{n-2}{2}\right)^2\right)\left(\mathrm{ch}\left(\frac{n-2}{2}t\right)u\right) \quad = \quad \left(\mathrm{ch}\left(\frac{n-2}{2}t\right)\partial_t^2 + (n-2)\mathrm{sh}\left(\frac{n-2}{2}t\right)\right)u$$

So we can conjugate the g_ε-Laplacian by a multiple of the function $\mathrm{ch}(t(n-2)/2)$ - in particular, of course, by u_ε - to obtain the following identity

$$\Delta_{g_\varepsilon} \quad = \quad u_\varepsilon^{-\frac{n+2}{n-2}}\,\mathcal{L}_\varepsilon\,(u_\varepsilon\,\cdot) \tag{5}$$

where

$$\mathcal{L}_\varepsilon \quad = \quad \partial_t^2 \;-\; \left(\frac{n-2}{2}\right)^2 \;+\; \Delta_{S^{n-1}}^{(\theta)} \;+\; u_\varepsilon^{\frac{4}{n-2}}\Delta_K^{(z)} \;+\; \mathcal{O}\left(|x|\right)\tilde{\Phi}(\nabla,\nabla^2)$$

where the linear second order differential operator $\tilde{\Phi}(\nabla,\nabla^2)$ enjoys similar properties as the operator Φ above. For $(n-2)/2 \leq \delta \leq 0$ we have that

$$\mathcal{L}_\varepsilon(\mathrm{ch}t)^\delta \quad = \quad \left[\delta^2 - \left(\frac{n-2}{2}\right)^2 + \mathcal{O}\left(|x|\right)\right](\mathrm{ch}t)^\delta + \left(\delta-\delta^2\right)(\mathrm{ch}t)^{\delta-2}$$

By our choice of the parameter δ we have immediately

$$\delta-\delta^2 \leq 0 \qquad \text{and} \qquad \delta^2 - \left(\frac{n-2}{2}\right)^2 \leq 0$$

In order to estimate the term $\mathcal{O}\left(|x|\right)$ let us take $\alpha > 0$ and let $\varepsilon_\alpha \in (0,1)$ be chosen so that $\log\varepsilon_\alpha + \alpha < 0$ or equivalently $\varepsilon_\alpha \mathrm{e}^\alpha < 1$, then it is easy to see that $|x| \leq \mathrm{e}^{-\alpha}$ for every $\varepsilon \in (0,\varepsilon_\alpha)$ and every $t \in [\log\varepsilon+\alpha, 0]$. Finally, by choosing $\alpha > 0$ such that

$$\mathrm{e}^{-\alpha} \quad \leq \quad -\frac{1}{2}\left(\delta^2 - \left(\frac{n-2}{2}\right)^2\right)$$

we obtain that, for every $\varepsilon \in (0,\varepsilon_\alpha)$ and for $t \in [\log\varepsilon+\alpha, 0]$

$$\mathcal{L}_\varepsilon(\mathrm{ch}t)^\delta \quad \leq \quad \frac{1}{2}\left(\delta^2 - \left(\frac{n-2}{2}\right)^2\right)(\mathrm{ch}t)^\delta$$

When $0 \leq \delta \leq (n-2)/2$ we use the function $\mathrm{ch}(\delta t)$ and we get

$$\begin{aligned} \mathcal{L}_\varepsilon \mathrm{ch}(\delta t) \quad &= \quad \left(\delta^2 - \left(\frac{n-2}{2}\right)^2 + \mathcal{O}\left(|x|\right)\right)\mathrm{ch}\delta t \\ &\leq \quad \frac{1}{2}\left(\delta^2 - \left(\frac{n-2}{2}\right)^2\right)\mathrm{ch}\delta t \end{aligned}$$

with similar restrictions on ε and t.

We define the function φ_δ by

$$\begin{aligned} \varphi_\delta \quad &= \quad u_\varepsilon^{-1}(\mathrm{ch}t)^\delta \qquad &\text{if} \quad \frac{n-2}{2} \leq \delta \leq 0 \\ \varphi_\delta \quad &= \quad u_\varepsilon^{-1}\mathrm{ch}\delta t \qquad &\text{if} \quad 0 \leq \delta \leq \frac{n-2}{2} \end{aligned}$$

and taking into account the conjugation (5) described above, we can state the following

Lemma 4.1. *Given $\delta \in (-\frac{n-2}{2}, \frac{n-2}{2})$ there exist a real number $\alpha = \alpha(n,\delta) > 0$ and a constant $C = C(n,\delta) \geq 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ we have*

$$\Delta_{g_\varepsilon} \varphi_\delta \quad \leq \quad -C u_\varepsilon^{-\frac{4}{n-2}} \varphi_\delta \tag{6}$$

in the set $T_\alpha^\varepsilon = \{\log \varepsilon + \alpha \leq t \leq -\log \varepsilon - \alpha\}$.

In particular the functions φ_δ can be used as barrier functions in the set $T_\alpha^\varepsilon = \{\log \varepsilon + \alpha \leq t \leq -\log \varepsilon - \alpha\}$.

4.2 Local *a priori* estimate using the maximum principle

We first provide a local *a priori* estimate for the g_ε-Laplacian, then we will observe that a similar estimate holds for the operator L_{g_ε}. This later estimate uses the scalar curvature estimate of the previous section since the term S_{g_ε} appears in the expression of L_{g_ε}.

Let us assume that v, f are bounded functions satisfying $\Delta_{g_\varepsilon} v = f$ in T_α^ε. The inequality found in Lemma 4.1 multiplied by a nonnegative real constant $a \geq 0$ yields

$$\Delta_{g_\varepsilon} (a\varphi_\delta - v) \quad \leq \quad -aC u_\varepsilon^{-\frac{4}{n-2}} \varphi_\delta - f$$

If we chose

$$a = C' \left(\sup_{T_\alpha^\varepsilon} \left| u_\varepsilon^{\frac{4}{n-2}} \varphi_\delta^{-1} f \right| \; + \; \sup_{\partial T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \right)$$

where $C' = \max\{1, C^{-1}\}$ and $\partial T_\alpha^\varepsilon = \{t = \pm \log \varepsilon \pm \alpha\}$, we obtain immediately

$$\begin{aligned} \Delta_{g_\varepsilon} (a\varphi_\delta - v) \quad &\leq \quad 0 \qquad \text{in} \quad T_\alpha^\varepsilon \\ a\varphi_\delta - v \quad &\geq \quad 0 \qquad \text{on} \quad \partial T_\alpha^\varepsilon \end{aligned}$$

Hence, by the maximum principle $a\varphi_\delta - v \geq 0$ on T_α^ε. In particular, we get

$$\sup_{T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \quad \leq \quad C' \left(\sup_{T_\alpha^\varepsilon} \left| u_\varepsilon^{\frac{4}{n-2}} \varphi_\delta^{-1} f \right| \; + \; \sup_{\partial T_\alpha^\varepsilon} \left| \varphi_\delta^{-1} v \right| \right)$$

In order to simplify the above expression, which is the estimate we were looking for, it is sufficient to replace u_ε by its expression and to observe that for every $\lambda \in \mathbb{R}$ there exist two constants $K_1(\lambda), K_2(\lambda) \geq 0$ such that

$$K_1(\lambda) \, (\mathrm{ch} t)^\lambda \leq \mathrm{ch} \lambda t \leq K_2(\lambda) \, (\mathrm{ch} t)^\lambda$$

for $t \in \mathbb{R}$.

Performing simple manipulations, the above estimate can be written as

$$\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right| \quad\leq\quad C_{n,\delta}\left(\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}f\right| \;+\; \sup_{\partial T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right|\right)$$

where $\psi_\varepsilon = \varepsilon\,\mathrm{ch}t$.

Now let us assume that $v, f \in \mathcal{C}^\infty(T_\alpha^\varepsilon)$ are functions verifying $L_{g_\varepsilon} v = f$. By the previous result we immediately have

$$\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right| \quad\leq\quad C_{n,\delta}\left(\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}f\right| \;+\; \sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}\,S_{g_\varepsilon}v\right| \;+\; \sup_{\partial T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right|\right)$$

Now let us look at the term $\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}\,S_{g_\varepsilon}v\right|$, which can be obviously written as $\left|\psi_\varepsilon^2 S_{g_\varepsilon}\right|\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right|$. The only term to control is the factor $\left|\psi_\varepsilon^2 S_{g_\varepsilon}\right|$, but thanks to the scalar curvature estimate we can say that, for a suitable constant $C'' > 0$

$$\left|\psi_\varepsilon^2 S_{g_\varepsilon}\right| \leq C''\left(\varepsilon + \mathrm{e}^{-\alpha}\right)$$

for all $\varepsilon \in (0, \varepsilon_\alpha)$. Hence, if $\alpha > 0$ is fixed large enough, we get

$$C_{n,\delta}\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}\,S_{g_\varepsilon}v\right| \quad\leq\quad \frac{1}{2}\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right|$$

Introducing this information back in the above estimate, we get

Proposition 4.2. *Given $\delta \in (-\frac{n-2}{2}, \frac{n-2}{2})$, there exist a real number $\alpha = \alpha(n,\delta) > 0$ and a constant $C_{n,\delta} \geq 0$ such that for all $\varepsilon \in (0, \varepsilon_\alpha)$ and all $v, f \in \mathcal{C}^0(T_\alpha^\varepsilon)$ satisfying $L_{g_\varepsilon} v = f$, the following estimate holds*

$$\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right| \quad\leq\quad C_{n,\delta}\left(\sup_{T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n+2}{2}-\delta}f\right| \;+\; \sup_{\partial T_\alpha^\varepsilon}\left|\psi_\varepsilon^{\frac{n-2}{2}-\delta}v\right|\right) \tag{7}$$

where $\psi_\varepsilon = \varepsilon\,\mathrm{ch}t$.

4.3 Global *a priori* estimate

Thanks to the previous local result, we will be able to prove a global *a priori* estimate. To introduce the result, we define a smooth function ψ_ε by

$$\psi_\varepsilon := \begin{cases} \varepsilon\mathrm{ch}t & \text{in } T_\alpha^\varepsilon \\ 1 & \text{in } M \setminus T_0^\varepsilon \end{cases}$$

where $T_\rho^\varepsilon := \{\log\varepsilon + \rho \leq t \leq -\log\varepsilon - \rho\}$, for $\rho > 0$ and ψ_ε interpolate smoothly between these definitions in $T_0^\varepsilon \setminus T_\alpha^\varepsilon$.

Proposition 4.3. *Let $M = M_1 \sharp_K M_2$ be the generalized connected sum obtained by removing an ε-tubular neighborhood V_i^ε of $\iota_i(K)$ from each M_i, $i = 1, 2$ and identifying the two boundaries. Suppose that both L_{g_1} and L_{g_2} have trivial kernels on M_1 and on M_2 respectively, then for every $\delta \in \left(-\frac{n-2}{2}, \frac{n-2}{2}\right)$ there exist a real number $\alpha = \alpha(n, \delta) > 0$ and a constant $C_{n,\delta} \geq 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ and every functions $v, f \in \mathcal{C}^0(M)$ satisfying $L_{g_\varepsilon} v = f$, the following estimate holds*

$$\sup_M \left| \psi_\varepsilon^{\frac{n-2}{2}-\delta} v \right| \quad \leq \quad C_{n,\delta} \left(\sup_M \left| \psi_\varepsilon^{\frac{n+2}{2}-\delta} f \right| \right) \tag{8}$$

The proof is by contradiction. Let us assume that the statement is false. Then for every $j \in \mathbb{N}$ we can find a triple $(\varepsilon_j, v_j, f_j)$ such that

1. $\varepsilon_j < \mathrm{e}^{-j}$
2. $L_{g_{\varepsilon_j}} v_j = f_j$
3. $\sup_M \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| = 1$
4. $\lim_{j\to\infty} \sup_M \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}} f_j \right| = 0$

For every $j \in \mathbb{N}$ we consider a point p_j such that $\left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta}(p_j) v_j(p_j) \right| = 1$, then (up to a subsequence) we have to distinguish two cases :

Case 1 $p_j \in M \setminus T_\alpha^{\varepsilon_j}$ for every $j \in \mathbb{N}$

Case 2 $p_j \in T_\alpha^{\varepsilon_j}$ for every $j \in \mathbb{N}$

Without loss of generality we can assume (up to a subsequence) that $p_j \in M_1 \setminus V_1^{\varepsilon_j}$, for all $j \in \mathbb{N}$, so, in the first case all the p_j's are in the compact set $Q_1^{\mathrm{e}^{-\alpha}} = M_1 \setminus V_1^{\mathrm{e}^{-\alpha}}$, then (up to a subsequence) they must converge to a point $p_\infty \in Q_1^{\mathrm{e}^{-\alpha}}$. We prove now that, for every compact set $Q^\sigma = Q_1^\sigma \cup Q_2^\sigma = (M_1 \setminus V_1^\sigma) \cup (M_2 \setminus V_2^\sigma)$, $\sigma > 0$, the sequence of functions $\{v_j\}_{j\in\mathbb{N}}$ converges (up to a subsequence) to a function v_∞ in $L^\infty(Q^\sigma)$. This will in particular imply that $|v_\infty(p_\infty)| > 0$.

In order to prove the uniform convergence of the v_j's on the compact Q^σ, we start by observing that

$$|v_j| \leq \left(\inf_{Q^\sigma} \left| \psi_j^{\frac{n-2}{2}-\delta} \right| \right)^{-1} \leq \frac{2}{\sigma}$$

and hence $\|v_j\|_{L^\infty(Q^\sigma)} \leq 2/\sigma$.

The next step is to get a $L^\infty(Q^\sigma)$- uniform bound for ∇v_j. To do that we need the following L^p-regularity result [3] for solutions of linear elliptic equations

Theorem 4.4. *Let be $L = a^{ij}\partial_{ij} + b^i\partial_i + c$, where the a, b, c's are functions defined on an open domain $\Omega \subset \mathbb{R}^m$, let be $1 < p < \infty$ and let be $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$. Moreover suppose that:*

1. *$a^{ij} \in \mathcal{C}^0(\Omega)$; $b^j, c \in L^\infty(\Omega)$; $f \in L^p(\Omega)$*
2. *There exist $\lambda, \Lambda > 0$ such that $|a^{ij}|, |b^j|, |c| \leq \Lambda$ and $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for every $\xi \in \mathbb{R}^n$*
3. *$Lu = f$*

then, for every $\Omega' \subset\subset \Omega$, the following estimate holds:

$$\|u\|_{W^{2,p}(\Omega')} \quad \leq \quad C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right)$$

for a suitable constant C.

This result can be restated in our context by saying:

Corollary 4.5. *Let be $\sigma > 0$ and suppose that the linear elliptic differential operator $L_g = \Delta_g + c$ is defined on a geodesic ball $B_{\sigma/2}$ of the Riemannian manifold (M, g), where c is a continuous bounded function on $B_{\sigma/2}$. Moreover let be $1 < p < \infty$ and let be $u \in W^{2,p}_{loc}(B_{\sigma/2}) \cap L^p(B_{\sigma/2})$, $f \in L^p(B_{\sigma/2})$ such that $L_g u = f$, then for every $0 < r < \sigma/2$ the following estimate holds*

$$\|u\|_{W^{2,p}(B_r)} \quad \leq \quad C\left(\|u\|_{L^p(B_{\sigma/2})} + \|f\|_{L^p(B_{\sigma/2})}\right)$$

for a suitable constant C (depending on σ).

In our case it is convenient to cover the compact set Q^σ by finitely many geodesic balls of radius $r = \sigma/4$. We can state

$$\begin{aligned}
\|v_j\|_{W^{2,p}(B_{\sigma/4})} &\leq C\left(\|v_j\|_{L^p(B_{\sigma/2})} + \|f_j\|_{L^p(B_{\sigma/2})}\right) \\
&\leq C'\left(\|v_j\|_{L^\infty(B_{\sigma/2})} + \|f_j\|_{L^\infty(B_{\sigma/2})}\right) \\
&\leq \frac{C''}{\sigma}
\end{aligned}$$

Thanks to Sobolev Embedding Theorem with $p > m/2$, $W^{2,p}$ is continuously embedded in L^∞, so $\|\nabla v_j\|_{L^\infty(B_{\sigma/4})} \leq C'''/\sigma$. Now, by Ascoli's Theorem, we conclude that (up to a subsequence) the sequence $\{v_j\}_{j\in\mathbb{N}}$ converges uniformly to a function v_∞ on every $B_{\sigma/4}$. Using a classical diagonal argument we have the convergence on each Q^σ.

To summarize, in the **Case 1**, we have found a subsequence such that $v_j \to v_\infty$ with respect to the L^∞-norm on any Q^σ, in particular $v_\infty \in \mathcal{C}^0(Q^\sigma)$, and, for $\sigma = \mathrm{e}^{-\alpha}$, we get $|v_\infty(p_\infty)| > 0$, as we have already remarked.

Now, let us consider **Case 2**. Since each p_j is in $T_\alpha^{\varepsilon_j}$, we can apply the local *a priori* estimate (7) obtained in the previous section to get

$$\begin{aligned} C_{n,\delta}^{-1} &\leq \sup_{T_\alpha^{\varepsilon_j}} \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}-\delta} f_j \right| + \sup_{\partial T_\alpha^{\varepsilon_j}} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| \\ &\leq \sup_{M} \left| \psi_{\varepsilon_j}^{\frac{n+2}{2}-\delta} f_j \right| + \max_{\partial V_1^{e^{-\alpha}}} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| + \max_{\partial V_2^{e^{-\alpha}}} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta} v_j \right| \end{aligned}$$

This shows that we can choose a sequence of points $q_j \in \partial V_1^{e^{-\alpha}} \cup \partial V_2^{e^{-\alpha}}$ such that

$$\lim_{j\to\infty} \left| \psi_{\varepsilon_j}^{\frac{n-2}{2}-\delta}(q_j) v_j(q_j) \right| = C_{n,\delta}^{-1}$$

In particular we have that $\lim_{j\to\infty} |v_j(q_j)| = 2C_{n,\delta}^{-1} e^{\alpha}$, then by using the L^∞-convergence to v_∞ on the compact set $Q^{e^{-\alpha}}$, it is easy to see that $|v_\infty(q_\infty)| > 0$, where $q_\infty \in \partial V_1^{e^{-\alpha}} \cup \partial V_2^{e^{-\alpha}}$ is the limit (up to a subsequence) of the sequence $\{q_j\}$.

Hence, in both the cases, we have found a point $P \in M \setminus T_\alpha^\varepsilon$ such that $v_\infty(P) \neq 0$. Without loss of generality we can suppose that $P \in M_1 \setminus \iota_1(K)$: if we prove that $L_{g_1} v_\infty = 0$ on M_1, then by the hypothesis on the kernel of L_{g_1}, v_∞ must be identically zero and we have a contradiction.

Hence, it remains to prove that v_∞ is in the kernel of L_{g_1}. This will be achieved in two steps. The first one amounts to say that $L_{g_1} v_\infty = 0$ on $M_1 \setminus \iota_1(K)$ in the sense of distributions, the second one amounts to estimate the growth of v_∞ near $\iota_1(K)$ and then to conclude by means of the following classical result.

Proposition 4.6. *Suppose that*

$$\begin{cases} L_{g_1} u &= 0 & \text{in}\, \mathcal{D}'(M_1 \setminus \iota_1(K)) \\ |u| &\leq C\, |\mathrm{d}_{g_1}(\cdot, \iota_1(K))|^{-\gamma} & \text{in}\, V_1^\rho \end{cases}$$

For $0 < \gamma < n-2$, a suitable real number $\rho > 0$ and a constant $C \geq 0$, then $u \in \mathcal{C}^\infty(M_1)$ and satisfies $L_{g_1} u = 0$ on M_1.

We choose $\varphi \in \mathcal{D}(M_1 \setminus \iota_1(K))$ and $\sigma > 0$ such that $\operatorname{supp}\varphi \subset Q^\sigma$. We claim that

$$\int_{M_1} v_\infty L_{g_1} \varphi \, \mathrm{d}vol_{g_1} = 0$$

This identity is obtained by taking the limit, as ε_j tends to 0 in the expression

$$\int_M v_j \, L_{g_{\varepsilon_j}} \varphi \, \mathrm{d}vol_{g_{\varepsilon_j}} = \int_M f_j \, \varphi \, \mathrm{d}vol_{g_{\varepsilon_j}}$$

Clearly, the right hand side of this expression tends to zero as ε_j tends to 0. As far as the right hand side is concerned g_{ε_j} converges (in $\mathcal{C}^2$ topology) to g_1

on Q^σ and hence $L_{g_{\varepsilon_j}}\varphi$ converges to $L_{g_1}\varphi$ in this set so that the left hand side converges to the required expression as ε_j tends to 0.

Finally we have to control the growth of v_∞ near $\iota_1(K)$. We remark that, on V_1^ρ

$$\frac{1}{2}|x| \le \psi_{\varepsilon_j} \le 2\,|x|$$

for every j, in particular

$$|x|^{\frac{n-2}{2}-\delta}|v_j| \le C$$

Hence

$$|v_\infty| \le C|x|^{\delta-\frac{n-2}{2}} = C|x|^{-\gamma}$$

where $\gamma = \frac{n-2}{2} - \delta$. Since $-\frac{n-2}{2} < \delta < \frac{n-2}{2}$, then $0 < \gamma < n-2$, as needed.

5 The nonlinear fixed point argument

We are now ready to solve equation (3). Observe that, as a consequence of the Proposition 4.3 , the operator L_{g_ε} is injective for sufficiently small ε. Since it is also self-adjoint, then it is invertible. Now we are looking for a function $v_\varepsilon \in L^\infty(M)$, $\|v\|_{L^\infty(M)} \le 1/2$ such that

$$v_\varepsilon = L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v_\varepsilon) \tag{9}$$

where

$$F_\varepsilon(v) := c_n(S - S_{g_\varepsilon}) + c_n(S - S_{g_\varepsilon})v + c_n\left((1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}v\right)$$

In other words we are looking for a fixed point for the operator $L_{g_\varepsilon}^{-1} \circ F_\varepsilon$.

We claim that, for a suitable choice of δ and for sufficiently small ε there exists a real number $r_\varepsilon > 0$ such that

$$\|v\|_{L^\infty(M)} \le r_\varepsilon \quad\Longrightarrow\quad \|L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v)\|_{L^\infty(M)} \le r_\varepsilon$$

Indeed, using the scalar curvature estimates it is easy to see that

$$\sup_M \left|\psi_\varepsilon^{\frac{n+2}{2}-\delta} F_\varepsilon(v)\right| \quad\le\quad C'\left(\varepsilon^{n-2} + \varepsilon^{\frac{n}{2}-\delta} + \|v\|^2_{L^\infty(M)}\right)$$

Now

$$\psi_\varepsilon^{\delta-\frac{n-2}{2}}\left|\varepsilon^{n-2} + \varepsilon^{\frac{n}{2}-\delta} + \|v\|^2_{L^\infty(M)}\right| \quad\le\quad C''\left(\varepsilon^{\frac{n-2}{2}+\delta} + \varepsilon + \varepsilon^{\delta-\frac{n-2}{2}}\|v\|^2_{L^\infty(M)}\right)$$

Therefore, using the estimate (8) and the hypothesis of the claim we get

$$\|L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v)\|_{L^\infty(M)} \quad\le\quad C'''\left(\varepsilon^{\frac{n-2}{2}+\delta} + \varepsilon + \varepsilon^{\delta-\frac{n-2}{2}} r_\varepsilon^2\right)$$

where $C''' = CC'C''$. To prove the claim it is sufficient to choose $r_\varepsilon > 0$ such that

$$\varepsilon^{\delta - \frac{n-2}{2}} r_\varepsilon^2 \leq r_\varepsilon/(2C''') \qquad \text{and} \qquad \varepsilon^{\frac{n-2}{2}+\delta} + \varepsilon \leq r_\varepsilon/(2C''')$$

The first condition is satisfied if we choose $r_\varepsilon = \varepsilon^{\frac{n-2}{2}-\delta}/(2C''')$. By this choice the second inequality becomes

$$\varepsilon^{2\delta} + \varepsilon^{\delta - \left(\frac{n-2}{2}-1\right)} \leq 1/(2C''')^2$$

Now it is clear that if $\max\{0, (n-2)/2 - 1\} < \delta < (n-2)/2$, then it is possible to find $\varepsilon_0 \in (0, \varepsilon_\alpha)$ such that the last inequality is verified for all $\varepsilon \in (0, \varepsilon_0)$. For those ε's, we can choose $r_\varepsilon = \varepsilon^{\frac{n-2}{2}-\delta}/(2C''')$ and the claim follows, hence

$$\|L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v)\|_{L^\infty(M)} \leq r_\varepsilon$$

It is easy to check that the mapping

$$v \in L^\infty(M) \longmapsto L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v) \in L^\infty(M)$$

is continuous and compact. This later property follows from the fact that the equation we want to solve is a semilinear equation and hence, if $v \in L^\infty(M)$ then $L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v) \in W^{2,p}(M)$ for all $p > 1$. The claim follows from the fact that the embedding $W^{2,p}(M) \longrightarrow L^\infty(M)$ is compact, provided $p > m/2$. Applying Schauder's fixed point Theorem yields the existence of a fixed point $v_\varepsilon \in L^\infty(M)$ to

$$v_\varepsilon = L_{g_\varepsilon}^{-1} \circ F_\varepsilon(v_\varepsilon)$$

which satisfies $\|v_\varepsilon\|_{L^\infty(M)} \leq r_\varepsilon$.

A priori the function v_ε is only bounded but, by a simple boot-strap argument (based on Corollary 4.5), one can easily checks that $v_\varepsilon \in \mathcal{C}^\infty(M)$.

Finally, observe that as $\varepsilon \to 0$, then $r_\varepsilon \to 0$ and consequently so does $\|v_\varepsilon\|_{L^\infty(M)}$. This shows that the conformal factor $u_\varepsilon = 1 + v_\varepsilon$ is as close to 1 as we want. This completes the proof of the main Theorem. The estimate in the statement of the Theorem follows at once from the definition of r_ε.

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